Lecture 3: The Discrete-Time Representative Agent Model

In this lecture I consider a discrete-time representative agent model with money in the utility function originally due to Brock (1974,1975). I begin, however, by considering the necessary and sufficient conditions for the household's optimisation problem in general in discrete-time representative agent models.

1 Necessary and Sufficient Conditions for Optimality

We begin with a useful property of concavity.

1.1 A useful property of concavity

Recall that a real-valued twice-differentiable function f on the interval S is concave on S if and only if $f''(x) \leq 0$ on S. Let f be a concave function on S and let x and x^* be any two elements of S. By the mean value theorem there exists an $\alpha \in [0,1]$ such that $f(x) - f(x^*) = f'(\alpha x + (1-\alpha)x^*)(x - x^*)$. Note that the derivative of the right-hand side of this equation with respect to α is $f''(\alpha x + (1-\alpha)x^*)(x - x^*)^2 \leq 0$. Thus, $f'(\alpha x + (1-\alpha)x^*)(x - x^*) \leq f'(x^*)(x - x^*)$. We have then that concavity implies that

$$f(x) - f(x^*) \leq f'(x^*)(x - x^*). \tag{1}$$

Now suppose that $f(x)$ is a twice-differentiable concave function on the interval S and that there exists $x^* \in S$ such that the first-order condition for a maximum, $f'(x^*) = 0$, is satisfied. We can use inequality (1) to prove that x^* maximises f on S. Let x be any other element of S. Then by inequality (1) and the first-order condition $f(x) - f(x^*) \leq f'(x^*)(x - x^*) = 0$. Thus, given that f is concave, the first-order condition is sufficient for a maximum.

1.2 Finite-horizon optimisation problems

1.2.1 A two-period problem

Consider the simple two-period optimisation problem:

$$\max u(c_0) + \beta u(c_1), \; 0 < \beta < 1 \text{ subject to}$$

$$
\begin{aligned}
c_0 + k_1 &= f(k_0) \\
c_1 + k_2 &= f(k_1) \\
c_0 &\geq 0,\ c_1 \geq 0,\ k_1 \geq 0,\ k_2 \geq 0,\ k_0 \text{ given.}
\end{aligned}
$$

Assume that the functions u and f are twice-differentiable concave functions with strictly positive first derivatives. We show that if there is a $(c_0^*, c_1^*, k_1^*, k_2^*)$ such that $c_0^* > 0$, $c_1^* > 0$, the budget constraints are satisfied, $k_2^* = 0$ and $\beta u'(c_1^*) f'(k_1^*) = u'(c_0^*)$ (the Euler equation) then this $(c_0^*, c_1^*, k_1^*, k_2^*)$ is an optimum.

Our proof involves demonstrating that utility is at least as great at $(c_0^*, c_1^*, k_1^*, k_2^*)$ as it is at any other admissible (c_0, c_1, k_1, k_2). Before proceeding note that it is admissible for consumption to be zero even though the utility function might not be defined at zero. The logarithmic utility function, for example, goes to minus infinity as consumption falls to zero. However, since the solution has strictly positive consumption it would always dominate an admissible (c_0, c_1, k_1, k_2) with zero consumption. So, we can restrict our comparison consumption and capital 4-tuples to ones with strictly positive consumption in both periods.

We have that $(c_0^*, c_1^*, k_1^*, k_2^*)$ is an optimum if

$$D := u(c_0) + \beta u(c_1) - u(c_0^*) - \beta u(c_1^*) \leq 0.$$

Using the concavity property in inequality (1), we have

$$D \leq u'(c_0^*)(c_0 - c_0^*) + \beta u'(c_1^*)(c_1 - c_1^*)$$

$$= u'(c_0^*)\left[f(k_0) - k_1 - f(k_0) + k_1^*\right] + \beta u'(c_1^*)\left[f(k_1) - k_2 - f(k_1^*) + k_2^*\right]$$

by the budget constraints

$$= -u'(c_0^*)(k_1^* - k_1) + \beta u'(c_1^*)\left[f(k_1) - f(k_1^*)\right] + \beta u'(c_1^*)(k_2^* - k_2)$$

because k_0 is given

$$\leq \left[\beta u'(c_1^*) f'(k_1^*) - u'(c_0^*)\right](k_1 - k_1^*) - \beta u'(c_1^*)(k_2 - k_2^*)$$

because f is concave

$$
\begin{aligned}
&= -\beta u'(c_1^*)\,(k_2 - k_2^*) \\
&= -\beta u'(c_1^*)k_2 \leq 0 \text{ because } \beta u'(c_1^*)k_2^* = 0.
\end{aligned}
$$

Thus, given our regularity assumptions and the initial condition, it is *sufficient* for a maximum that the budget constraints, the Euler equation $\beta u'(c_1)f'(k_1) = u'(c_0)$, and the *transversality condition* $\beta u'(c_1)k_2 = 0$ hold.

Clearly satisfying the Euler equation is not necessary for an optimum. There might not exist a solution; instead we might have a corner solution. It seems clear, however, that satisfying the transversality condition is necessary. If the marginal utility of consumption is positive in period one it cannot be part of an optimum to give up consumption to accumulate useless capital.

1.2.2 A T-period problem

We can extend the model to T periods:

$$
\max \sum_{t=0}^{T} \beta^t u(c_t),\ 0 < \beta < 1 \text{ subject to}
$$

$$
\begin{aligned}
c_t + k_{t+1} &= f(k_t)\,,\ t = 0, 1, ..., T-1 \\
c_t &\geq 0,\ k_{t+1} \geq 0,\ t = 0, 1, ...T,\ k_0 \text{ given.}
\end{aligned}
$$

The Euler equations are:

$$
\beta u'(c_{t+1}^*)f'\left(k_{t+1}^*\right) = u'(c_t^*),\ \ t = 0, ...T-1
$$

and the transversality condition is

$$
\beta^T u'(c_T^*)k_{T+1}^* = 0.
$$

Using the same technique as in the 2-period case we have

$$
\begin{aligned}
D \quad : & = \sum_{t=0}^{T} \beta^t \left[u(c_t) - u(c_t^*)\right] \\
\leq & \sum_{t=0}^{T} \beta^t u'(c_t^*) \left(c_t - c_t^*\right) \text{ by concavity of } u \\
= & \sum_{t=0}^{T} \beta^t u'(c_t^*) \left[f\left(k_t\right) - k_{t+1} - f\left(k_t^*\right) + k_{t+1}^*\right] \text{ by the budget constraints} \\
\leq & \sum_{t=0}^{T} \beta^t u'(c_t^*) \left[f'\left(k_t^*\right)\left(k_t - k_t^*\right) - \left(k_{t+1} - k_{t+1}^*\right)\right] \text{ by the concavity of } f \\
= & \sum_{t=1}^{T} \beta^t u'(c_t^*) f'\left(k_t^*\right)\left(k_t - k_t^*\right) - \sum_{t=0}^{T} \beta^t u'(c_t^*)\left(k_{t+1} - k_{t+1}^*\right) \\
& \text{ by the initial conditions} \\
= & \sum_{t=1}^{T} \beta^{t-1} u'(c_{t-1}^*)\left(k_t - k_t^*\right) - \sum_{t=0}^{T} \beta^t u'(c_t^*)\left(k_{t+1} - k_{t+1}^*\right) \\
& \text{by the Euler equations} \\
= & -\beta^T u'(c_T^*)\left(k_{T+1} - k_{T+1}^*\right) \leq \beta^T u'(c_T^*) k_{T+1}^* = 0.
\end{aligned}
$$

Thus, given our regularity assumptions and the initial condition, it is *sufficient* for a maximum that the budget constraints, the Euler equation and the *transversality condition* hold. If the marginal utility of consumption is positive in period T it cannot be part of an optimum to give up consumption to accumulate useless capital. Thus, the transversality condition is also necessary for a maximum. If we impose Inada conditions that rule out corner solutions, then satisfying the Euler equation and the transversality condition will be both necessary and sufficient.

1.3 Infinite-Horizon Problem

Consider the infinite-horizon problem:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t), \; 0 < \beta < 1 \text{ subject to}$$

$$
\begin{aligned}
c_t + k_{t+1} & = f\left(k_t\right), t \in \mathbb{Z}_+ \\
c_t & \geq 0, \, k_{t+1} \geq 0, \, t \in \mathbb{Z}_+, \, k_0 \text{ given.}
\end{aligned}
$$

The Euler equations are

$$\beta u'(c_{t+1}^*) f'\left(k_{t+1}^*\right) = u'(c_t^*), t \in \mathbb{Z}_+$$

and the transversality condition is

$$\lim_{t\to\infty} \beta^t u'(c_t^*) k_{t+1}^* = 0.$$

We can make a comparison solution argument here too. A technical detail is that if the utility function is bounded from below and the comparison consumption goes to zero over time the sum of discounted utility for the comparison allocation may not converge. So we write

$$D \equiv \lim \inf_{T\to\infty} \sum_{t=0}^{T} \beta^t \left[u(c_t) - u(c_t^*)\right]$$

and proceed as before. We will find that given the budget constraints and the initial condition, the Euler equation and the transversality condition are sufficient for optimality. But, now it is not quite so clear that the transversality condition is necessary.

That the transversality condition is a necessary condition in problems similar to this one was first proved by Weitzman (1973). His proof, however, requires the strong assumption that the utility function is bounded and so it is not applicable to the logarithmic case or to any utility function with the form $c^{1-\sigma}/\left(1-\sigma\right)$, where $\sigma > 1$. Ekeland and Scheinkman (1986) showed that under certain assumptions, the transversality condition is also necessary for unbounded utility functions. Both Weitzman (1973) and Ekeland and Scheinkman (1986) are difficult papers, to say the least. Kamihigashi (2000) relaxes some of Ekeland and Scheinkman's conditions and his proof is beautifully simple. It does however require that the sequence $\left\{\beta^t u\left(c_t\right)\right\}$ be summable (that is the sum of absolute values converges) at an optimum. This is slightly naughty because it is not an assumption on the fundamentals of the model. He also requires that there exists a constant θ and a summable sequence $\{b_t\}$ such that $u'\left(c_t\right) c_t \leq \theta u\left(c_t\right) + b_t$. This may or may not be reasonable, but lacks intuitive appeal (at least to me).

The idea behind his proof is as follows. Suppose that $\left\{c_t^*, k_{t+1}^*\right\}$ is optimal and consider a comparison allocation

$$\begin{aligned}
(c_t, k_{t+1}) &= \left(c_t^*, k_{t+1}^*\right), \ t = 0, ..., T-1 \\
(c_T, k_{T+1}) &= \left(c_T^* + (1-\lambda)\, k_{T+1}, \lambda k_{T+1}^*\right), \ t = T \\
(c_t, k_{t+1}) &= \left(\lambda c_t^*, \lambda k_{t+1}^*\right), \ t = T+1, ... ,
\end{aligned}$$

where $\lambda \in (0, 1)$. This comparison allocation is possible because the production function is concave. Because

$\left\{c_t^*, k_{t+1}^*\right\}$ is optimal

$$\begin{aligned}
\lim \inf_{T\to\infty} \sum_{t=0}^{T} \beta^t \left[u(c_t) - u(c_t^*)\right] &\leq 0 \Rightarrow \\
\beta^T \left[u\left(c_T^* + (1-\lambda)\, k_{T+1}\right) - u(c_T^*)\right] + \lim \inf_{S\to\infty} \sum_{t=T+1}^{S} \beta^t \left[u(\lambda c_t^*) - u(c_t^*)\right] &\leq 0 \Rightarrow
\end{aligned}$$

$$\frac{\beta^T \left[u\left(c_T^* + (1-\lambda)\, k_{T+1}\right) - u(c_T^*)\right]}{1-\lambda} \leq \lim \sup_{S\to\infty} \sum_{t=T+1}^{S} \beta^t \frac{u(c_t^*) - u(\lambda c_t^*)}{1-\lambda}.$$

Kamihigashi demonstrates that his assumptions imply that the right-hand side equals zero as $T \to \infty$. (This is not trivial but does not require any advanced math.) Then using the definition of a derivative we have that as $\lambda \to 1$

$$\lim_{T\to\infty} \beta^T u'\left(c_T^*\right) k_{T+1} \leq 0.$$

To summarise these results about transversality conditions. The transversality condition is that residual term that you need to have be non-negative when you do the standard proof that the constraints, the Euler equations and transversality condition are sufficient. Given certain assumptions, this transversality condition is also necessary for an optimum. If the Inada conditions are satisfied then under these assumptions the constraint, Euler equations and the transversality condition are necessary and sufficient for an optimum.

2 Brock's (1974,1975) Model

2.1 The households

The economy is inhabited by a representative household and its government. Each period, the household receives an exogenous endowment of the single perishable consumption good and it pays a lump-sum tax. It consumes the good and saves money. Utility is given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t, m_t),\ 0 < \beta < 1,$$

where $c_t \geq 0$ is time-t consumption and $m_t \geq 0$ is the household's end-of-period-t demand for real money balances to be carried into period $t+1$.

The household maximises its utility subject to the sequence of within-period budget constraints

$$m_t = y - \tau_t - c_t + \left(P_{t-1}/P_t\right) m_{t-1},\ t \in \mathbb{Z}_+, \tag{2}$$

where $y > 0$ is the constant per-period endowment, $\tau_t < y$ is the period-t real lump-sum tax and P_t is the time-t price of the good. Attention is restricted to outcomes where $1/P_t > 0$ for every $t \in \mathbb{Z}_+$. There is, however, always a non-monetary equilibrium where $1/P_t = 0$ for every $t \in \mathbb{Z}_+$. In this outcome, money is not held and the household consumes its after-tax endowment each period.

It is assumed that u is strictly increasing in its first argument, weakly increasing in its second argument, concave and continuously differentiable on $\mathbb{R}^2_{++}$. It is also assumed that Kamihigashi's assumptions hold so that the transversality condition is necessary. It is assumed that $u_c(c, m) \to \infty$ *as* $c \searrow 0$, $u_m(c, m) - u_c(c, m) \to \infty$ *as* $m \searrow 0$ and that there exists $\bar{u} \in \mathbb{R}_{++}$ such that $\lim_{m\to\infty} u_c(c, m) = \bar{u}$..

There are two possible cases for the marginal utility of real balances. Either the marginal utility it is strictly positive and goes to zero as real balances go to infinity or there is a satiation point (which can depend on c) $\hat{m}(c)$ such that $u_m(c, m) > (=)\ 0$ if $m < (\geq)\ \hat{m}(c)$.

The Euler equations are

$$u_c(c_t, m_t) = u_m(c_t, m_t) + (\beta P_t/P_{t+1})\, u_c(c_{t+1}, m_{t+1}),\ \ t \in \mathbb{Z}_+ \tag{3}$$

and the transversality condition is

$$\lim_{t\to\infty} \beta^t \left[u_c(c_t, m_t) - u_m(c_t, m_t)\right] m_t \leq 0. \tag{4}$$

The Euler equation (3) has the following interpretation. The household is indifferent between a (small) one-unit increase in period-t consumption, which yields utility of $u_c(c_t, m_t)$, and foregoing this consumption and acquiring a one-unit increase in period-t real balances, which yield current utility of $u_m(c_t, m_t)$ and which can be traded next period for P_t/P_{t+1} units of the consumption good which yields a discounted utility of $(\beta P_t/P_{t+1})\, u_c(c_{t+1}, m_{t+1})$.

The transversality condition is the analogue of the period-T complementary slackness condition in a T-period finite-horizon problem. This complementary slackness condition states that either $\beta^T [u_c(c_T, m_T) - u_m(c_T, m)] = 0$ or $m_T = 0$. Given our Inada conditions, $m_T > 0$ and the household is willing to hold real balances only up to the point where the marginal utility gain from the current liquidity services of money equals the marginal utility loss from decreased current consumption. In this infinite-horizon problem equation (4) implies that either the optimal value of the state variable, m_t, goes to zero as time goes to infinity or that its marginal contribution to the maximized value of the objective function, $\beta^t [u_c(c_t, m_t) - u_m(c_t, m_t)]$ becomes non-positive.

Kamigashi's results imply that the transversality condition is necessary. Brock (1974) demonstrates that the Inada conditions ensure that the solution is interior and the Euler equations are necessary as well. If the Euler equations (3) are necessary then substituting them into the transversality condition (4) yields an alternative specification of the transversality condition (which is how it was written in the last lecture):

$$\lim_{t\to\infty} \beta^t u_c(c_t, m_t) m_t = 0. \tag{5}$$

2.2 The government

The government's within-period budget constraint, assumed to hold with equality, is

$$m_t = g - \tau_t + (P_{t-1}/P_t)\, m_{t-1},\ t \in \mathbb{Z}_+, \tag{6}$$

where $g \in [0, y)$ is the constant per-period public spending. Let M_t be the time-t money supply. It is assumed that the money supply grows at a constant proportional rate: $M_{t+1}/M_t = \mu > 0,\ t \in \mathbb{Z}_+$.

2.3 Equilibrium

The household budget constraint (2) and the government budget constraint (6) imply the (non-independent) economy-wide resource constraint

$$c_t = c \equiv y - g,\ t \in \mathbb{Z}_+. \tag{7}$$

Substituting equation (7) and the money market clearing condition into the Euler equation (3) and the transversality condition (5) yields

$$\beta u_c(c, m_{t+1}) m_{t+1} = \mu[u_c(c, m_t) - u_m(c, m_t)] m_t,\ t \in \mathbb{Z}_+ \tag{8}$$

$$\lim_{t\to\infty} \beta^t u_c(c, m_t) m_t = 0. \tag{9}$$

There are two potential types of equilibria. First, given the constant fundamentals (y, g, μ), there is a *fundamental* (or *Markov* or *minimal-state-variable*) equilibrium where $m_t = m > 0$ for every $t \in \mathbb{Z}_+$. Constant real balances clearly satisfy the transversality condition (9). By equilibrium condition (8) such an equilibrium has

$$\mu u_m(c, m) = (\mu - \beta) u_c(c, m). \tag{10}$$

If $\mu < \beta$ or if $\mu = \beta$ and there is no satiation point for real balances then no fundamental equilibrium exists. If $\mu = \beta$ and there is satiation in real balances then any m that is at least as great as the satiation point $\hat{m}(c)$ satisfies equation (10). Such an outcome is a Friedman (1969) Optimal Quantity of Money equilibrium. If $\mu > \beta$, then the Inada conditions imply that at least one fundamental equilibrium exists. For this case, the additional restriction that real balances are a normal good ensures uniqueness.

In addition to fundamental equilibria, there can be a variety of non-fundamental (or non-stationary) equilibria. An equilibrium can be stable, with monotonic or cyclical convergence; it can be unstable, with either monotonic or cyclical divergence; there can be limit cycles and there can be chaotic behaviour. (See Matsuyama (1991). Azariadis's (1993) textbook is good for learning about exotic dynamics.) We are interested in equilibria where real balances go to infinity; such equilibria are called *deflationary bubbles*.

3 Deflationary bubbles

This section considers the existence of deflationary bubbles.

3.1 The definition of a deflationary bubble

Economists have many different definitions of bubbles, depending on the scenario under consideration. Here there are equilibria which depend solely on the fundamentals (and, hence, are not time varying) and equilibria which depend on time as well as on the fundamentals. Of the equilibria which depend on time as well as on the fundamentals, the ones that go to infinity over time are defined to be deflationary bubbles.

Note that this definition does not imply that any equilibrium sequence of prices that goes to zero must be a deflationary bubble or that all deflationary bubbles must have the price level going to zero. When the nominal money stock is falling, then a fundamental equilibrium has $P_{t+1}/P_t = M_{t+1}/M_t = \mu < 1$ and the price level goes to zero over time. When the nominal money stock is rising, a deflationary bubble has $P_{t+1}/P_t = \mu m_t/m_{t+1}$ and can be associated with rising prices if real balances are rising at a rate less than μ. Along such a path however, inflation will be less than the associated fundamental equilibrium's inflation rate of μ.

3.2 The existence of deflationary bubbles

3.2.1 The case of $\mu > 1$

Suppose that $\mu > 1$ and conditions are such that the transversality condition is necessary. Then no deflationary bubbles exist. To see this let $x_t \equiv u_c(c, m_t)m_t > 0$. By equilibrium condition (8), $x_{t+1}/x_t = (\mu/\beta)\,[1 - u_m(c, m_t)/u_c(c, m_t)]$, $t \in \mathbb{Z}_+$. By the Inada conditions, $x_{t+1}/x_t \to \mu/\beta$ as $m_t \to \infty$. Thus, $\forall \epsilon > 0, \exists T \in \mathbb{Z}_+$

such that $x_{T+t+1}/x_{T+t} > \mu/\beta - \epsilon$, $t \in \mathbb{Z}_+$. Let $\epsilon = (\mu - 1)/\beta$. Then $\beta x_{T+t+1} > x_{T+t}$, $t \in \mathbb{Z}_+$. Hence, $\beta^{T+t} x_{T+t}$ cannot go to zero as $t \to \infty$ and the transversality condition (9) is violated.

3.2.2 The case of $\mu = 1$

When $\mu = 1$, there exist pathological cases where the transversality condition is satisfied and deflationary bubbles exist. Obstfeld and Rogoff (1986) provide an example (suggested by Guillermo Calvo and Roque Fernandez). The utility function is separable and has the property that the marginal utility of money is $1/\ln(m)$ for m large. If initial real balances exceed the steady state then the sequence, $\{m_t\}$ that satisfies equilibrium condition (8) also satisfies the transversality condition (9) and has $m_t \to \infty$.

3.2.3 The case of $\beta < \mu < 1$

If $\beta < \mu < 1$ and $\{m_t\}$ satisfies equilibrium condition (8) then $\{m_t\}$ is an equilibrium sequence of real balances. To see this, if $\{m_t\}$ satisfies equilibrium condition (8) then $x_{t+1}/x_t \leq \mu/\beta, t \in \mathbb{Z}_+$. Thus, $\beta^t x_{T+t} \leq \mu^t x_T \to 0$ as $t \to \infty$, $T \in \mathbb{Z}_+$. Thus transversality condtion (9) is satisfied.

When $\beta < \mu < 1$ it is easy to find examples of deflationary bubble equilibria. See the homework assignment.

3.2.4 The case of $\mu \leq \beta$

When $\mu \leq \beta$ deflationary bubbles cannot exist. This a consequence of the Euler equilibrium condition (8), rather than the transversality condition (9). By equilibrium condition (8), $m_{t+1} = (\mu/\beta)\left[u_c(c, m_t)/u_c(c, m_{t+1}) - u_m(c, m_t)/u\right.$ $t \in \mathbb{Z}_+$. Thus, $m_{t+1} \leq (\mu/\beta)\left[u_c(c, m_t)/u_c(c, m_{t+1})\right] m_t$, $t \in \mathbb{Z}_+$ and, hence, $m_t \leq (\mu/\beta)^t \left[u_c(c, m_0)/u_c(c, m_t)\right] m_0$. Thus, $\lim_{t\to\infty} m_t \leq \lim_{t\to\infty} (\mu/\beta)^t \left[u_c(c, m_0)/u_c(c, m_t)\right] m_0 \leq \left[u_c(c, m_0)/\bar{u}\right] m_0 < \infty$.

3.3 The relationship between the transversality condition and the "no-bubble" boundary condition

Turning to a different scenario, consider the market for a particular company's stock in a model without money in the utility function. Under certainty the household's Euler equation corresponding to that stock says that $p_t u'(c_t) = \beta (p_{t+1} + d_{t+1}) u'(c_{t+1})$, $0 < \beta < 1$,where u is the within-period utility function and c_t, p_t and d_t are the time-t consumption demand, stock price (in terms of the consumption good) and (exogenous) dividend, respectively. Suppose that, as in the previous model, $c_t = c \equiv y - g$, $t \in \mathbb{Z}_+$. Then, solving the Euler equation forward would yield $p_t = \sum_{s=1}^{\infty} \beta^s d_{t+1} + \lim_{T\to\infty} \beta^T p_{t+T}$. Thus, the stock price consists of a term $F_t \equiv \sum_{s=1}^{\infty} \beta^s d_{t+1}$, which depends on the fundamentals (that is, the dividends), and a term $C_t \equiv \lim_{T\to\infty} \beta^T p_{t+T}$.

This latter term may be strictly positive if investors have self-fulfilling expectations that the price will rise by more than is justified by the fundamentals. Alternatively, this term may be written as $C_t = k\beta^t$, where $k \geq 0$. Solutions where $k > 0$ and, hence $C_t \neq 0$ are often referred to as *rational* or *equilibrium bubbles.* They might be viewed as unlikely or not "sensible" as they are not Markov or "minimal-state-variable" solutions in McCallum's (1983) sense as they depend on an extraneous variable: calendar time. In theoretical models it is typical to impose the boundary condition $\lim_{T\to\infty} \beta^T p_{t+T} = 0$ to rule out such equilibria. In empirical models, deviations between p_t and the fundamental component, F_t, are often referred to as a *bubble* and researchers often test for the existence of a bubble by testing whether the price can be explained by the fundamentals: in this example, this would be testing whether $p_t = F_t$.

The boundary condition ruling out bubble equilibria *looks* like a transversality condition and some researchers, for example Froot and Obstfeld (1991), call this condition a transversality condition. However, it is not related to the transversality condition which, under certain assumptions, is necessary and sufficient for household optimality. In the model of stock prices this transversality condition would be $\lim_{T\to\infty} \beta^T u'(c_T) p_T s_T \leq 0$, where s_t is the household's time-t holdings of the stock.

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